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LINEAR ORDINARY DIFFERENTIAL
OPERATORS OF THE SECOND ORDER

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LINEAR ORDINARY DIFFERENTIAL OPERATORS OF THE SECOND ORDER

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ABSTRACT

A study is made of the operator $L[y] = -y''+q(x)y = 0$, where y is in $L_2(0,\infty)$ and satisfies the boundary conditions $y(0) = 0$ and $\lim_{x\to\infty} (\alpha y + \beta y') = 0$, where α/β need not be real. For q(x) real, two situations arise depending on q(x). In the limit point case y is always independent of α/β, L is self-adjoint and has a real spectrum. In the limit circle case L has a pure point spectrum which lies in the upper half plane if $\mathrm{Im}[\alpha/\beta] > 0$ and in the lower half plane if $\mathrm{Im}[\alpha/\beta] < 0$. An elementary divisor theory for operators with completely continuous resolvents is considered. For q(x) complex, with $|\mathrm{Im}[q(x)]|$ bounded, the operator L is treated by a perturbation procedure starting with the operator L_0 in which $q_0(x) = \mathrm{Re}[q(x)]$. Representation theorems for some classes of operators are also considered.

LINEAR ORDINARY DIFFERENTIAL OPERATORS OF THE SECOND ORDER

By R.S. Phillips

1. Introduction

This paper* contains some exploratory material on the second order differential operator

$$(1)\qquad \begin{aligned} &L(y) = -y'' + q(x)y && 0 < x < \infty \\ &y(0) = 0 \\ &\lim_{x\to\infty} |\alpha y(x) + \beta y'(x)| = 0 \end{aligned}$$

where $q(x)$ is, say, piecewise continuous on $(0, \infty)$ but not necessarily real, and where the ratio α/β need not be real. We are interested only in $y(x)$ in $L_2(0, \infty)$. The boundary condition at $x = 0$ has been chosen as a matter of convenience; any other real boundary condition at $x = 0$ would do just as well.

Such operators occur in numerous problems in electromagnetic theory when one attempts to solve the problem by the method of separation of variables. A complex valued $q(x)$ corresponds to a conducting medium, whereas a non-real α/β may correspond, for instance, to a Sommerfeld propagation condition at infinity. A number of such problems relating to electromagnetic radiation and diffraction in spherically stratified regions are treated in a report by N. Marcuvitz [4]. The need for extending the classical results on the Sturm-Liouville problem to differential operators of the above type (1) has also been very clearly brought out by Marcuvitz.

A precise meaning will be assigned to the operator (1) in section two for the case when $q(x)$ is real valued, and this will later (section four) be extended to complex $q(x)$. For real $q(x)$ two situations arise depending on the function $q(x)$. In the limit point case there is only one possible interpretation for L and that is independent of α/β -- L is self-adjoint with a real spectrum. In the limit circle case, L has a pure point spectrum which lies in the upper half-plane for $\operatorname{Im}[\alpha/\beta] < 0$ and in the lower half-plane for $\operatorname{Im}[\alpha/\beta] > 0$. We show by example that in this case the spectrum need not be confined to a strip parallel to the real axis.

A rather large class of these operators (for instance, all operators of the limit circle type) have completely continuous resolvents. Such operators have pure point spectra and if their spectrum contains an infinite set of points, these points have no finite point of condensation. In section three we develop an elementary

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* The author is indebted to Professor Nathan Marcuvitz for many of the ideas which are developed in section two of this paper.

divisor theory for such operators.

We next treat the operator L for complex $q(x)$. However, we limit ourselves to the case $|\text{Im}|q(x)||$ bounded. L is then the sum of the closed linear operator L_o (with $q_o(x) = R_e|q(x)|$ and the bounded operator $A(y) = i\ \text{Im}|q(x)|\ y(x)$. As such L may be defined in a straightforward fashion as $L_o + A$ and treated as a perturbation of the operator L_o. We show, in fact, that if L_o is self-adjoint, then the spectrum of L lies in the strip $|\text{Im}(\lambda)| \leq ||A||$; whereas if the spectrum of L_o lies in, say, the lower half-plane, then the spectrum of L lies in the half-plane $\text{Im}(\lambda) \leq ||A||$. It is also shown that if the resolvent of L_o is completely continuous, then so is that of L.

Finally in section five we consider representation theorems. If $||R(\lambda,L)|| \leq (\text{Im}(\lambda) - \omega)^{-1}$ for $\text{Im}(\lambda) \geq \omega$, as is often the case with operators of the type we are considering, then

$$y = \frac{-1}{\pi i} \int_{i\gamma - \infty}^{i\gamma + \infty} R(\lambda,L) y \, d\lambda \qquad (\gamma > \omega)$$

for y in the domain of L. The integral converges in the mean. Further, if L has a pure point spectrum of a rather limited type we are able to establish an expansion theorem, again for y in the domain of L.

2. The operator L when q(x) is real.

We shall now give a precise meaning to the operator L for the case when $q(x)$ is real valued. It is clear that $-y'' + q(x) y$ will be well defined if and only if $y(x)$ and $y'(x)$ are absolutely continuous and $y(x)$ and $-y'' + q(x) y$ both belong to $L_2(0,\infty)$. We wish to further delimit the domain D of L so that for some λ a linear bounded operator $R(\lambda,L)$ in $L_2(0,\infty)$ exists satisfying the conditions

(2) $\quad (\lambda I - L) R(\lambda,L) y = y \qquad$ for all $y \in L_2(0,\infty)$

(3) $\quad R(\lambda,L)(\lambda I - L) y = y \qquad$ for all $y \in D$.

Such an operator $R(\lambda,L)$ is called the resolvent of L. We do not claim that every operator must have a resolvent. We shall simply limit our considerations to such operators. For one thing such operators are closed. For another, the spectrum, $\sigma(L)$, of an operator L is defined in terms of $R(\lambda,L)$ as the set of points λ where no such bounded operator exists. The complementary set is called the resolvent set, $\rho(L)$. Most important, of course, is the fact that the bounded resolvent is a very useful tool in the investigation of linear operators.

The resolvent is essentially the inverse operator for $\lambda I - L$. It is clear from (2) that the range of the resolvent is contained in the domain D of L and from (3) that this range contains D. Hence the range of $R(\lambda, L)$ is precisely D, which is independent of λ. It further follows from (2) that $R(\lambda, L)f$ is a solution of

$$\lambda y + y'' - q(x)\, y = f. \tag{4}$$

In order that the resolvent be well defined, it is necessary to pick out a single such solution. This is accomplished by properly delimiting D and insisting that the solution of (4) be chosen from D. In order to obtain any sort of representation theory it is necessary that D be dense in $L_2\,(0, \infty)$.

There is no difficulty at the regular end, $x = 0$, in delimiting D. Here we shall insist that $y(0) = 0$. This is a meaningful condition since, as we have already noted, any y in D must be absolutely continuous (or differ from an absolutely continuous function on a set of measure zero, in which case we may use the absolutely continuous element of this class in defining the boundary value).

The condition at infinity can not be imposed in such a straightforward fashion. It will be necessary to go through the Weyl type limiting argument. We first consider the corresponding operator for the finite interval (0,b), namely

$$\begin{aligned} L_b[y] &= -y'' + q(x)\, y \qquad\qquad 0 \le x \le b \\ y(0) &= 0 \\ \alpha y(b) + \beta y'(b) &= 0\,. \end{aligned} \tag{5}$$

The domain D_b of L_b is the set of functions $y(x)$ on (0,b) such that y and y' are absolutely continuous, y and $-y'' + q(x)\, y$ belong to $L_2(0,b)$, $y(0) = 0$, and $\alpha\, y(b) + \beta\, y'(b) = 0$. Let $\theta(x,\lambda)$ and $\phi(x,\lambda)$ be solutions of

$$\lambda y + y'' - q(x)\, y = 0 \tag{6}$$

satisfying

$$\begin{aligned} \phi(0,\lambda) &= 0 \qquad & \theta(0,\lambda) &= -1 \\ \phi'(0,\lambda) &= 1 & \theta'(0,\lambda) &= 0\,. \end{aligned} \tag{7}$$

These solutions will then be analytic in λ. Let $\ell(\lambda, b)$ be such that $\psi(x,\lambda) = \theta(x,\lambda) + \ell(\lambda,b)\, \phi(x,\lambda)$ satisfies $\alpha\psi(b,\lambda) + \beta\psi(b,\lambda) = 0$. Then, as is well known, the resolvent of L_b exists for all λ for which $\ell(\lambda,b)$ is finite and is given by

$$R(\lambda, L_b)f = \psi(x,\lambda) \int_0^x \phi(y,\lambda)\, f(y)dy + \phi(x,\lambda) \int_x^b \psi(y,\lambda) f(y) dy. \tag{8}$$

Now $\ell(\lambda,b)$ is given by

$$(9)\quad \ell(\lambda,b) = -\frac{\alpha\theta(b,\lambda)+\beta\theta'(b,\lambda)}{\alpha\phi(b,\lambda)+\beta\phi'(b,\lambda)} = -\frac{\theta(b,\lambda)z+\theta'(b,\lambda)}{\phi(b,\lambda)z+\phi'(b,\lambda)} \equiv \ell(\lambda,b,z)$$

where $z = \alpha/\beta$. Thus $\ell(\lambda,b,z)$ is a fractional linear transformation of the z-plane onto the ℓ-plane, depending on boundary conditions at $x = b$. Since $\phi(b,\lambda)\theta'(b,\lambda) - \theta(b,\lambda)\phi'(b,\lambda) = 1$, the transformation is non-degenerate and takes the closed z-plane onto the closed ℓ-plane in a 1-1 fashion. Following Weyl, we know that as z traces out the real axis, ℓ describes a circle C_b. For $\mathrm{Im}(\lambda) > 0$ the lower half of the z-plane goes into the interior of C_b; whereas if $\mathrm{Im}(\lambda) < 0$, the upper half of the z-plane goes into the interior of C_b. Further, for $0 < b' < b$, $C_b \subset C_{b'}$. Hence as $b \to \infty$ the circles converge either to a limit circle C_∞ or to a limit point. For $\mathrm{Im}(\lambda) > 0$ and z fixed or a function of b --- in either case remaining in the lower half-plane --- the functions of λ, $\ell(\lambda,b,z)$, form a normal family of functions. Hence given any sequence $b_n \to \infty$, there exists a subsequence b_{n_k} such that the $\ell(\lambda,b_{n_k},z)$ converge uniformly in every compact subset of $\mathrm{Im}(\lambda) > 0$ to a function $m(\lambda)$ regular in the upper half λ-plane. Again by the Weyl theory

$$(10)\quad \int_0^\infty |\theta(x,\lambda) + m(\lambda)\,\phi(x,\lambda)|^2\,dx \leq -\frac{\mathrm{Im}[m(\lambda)]}{\mathrm{Im}(\lambda)} .$$

Thus $\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\,\phi(x,\lambda) \in L_2(0,\infty)$ for $\mathrm{Im}(\lambda) > 0$. A similar situation occurs for $\mathrm{Im}(\lambda) < 0$ and $\mathrm{Im}(z) > 0$.

In the limit point case with $\mathrm{Im}(\lambda) > 0$, the entire lower half z-plane goes into the limit point (or for $\mathrm{Im}(\lambda) < 0$ the upper-half z-plane goes into the limit point). The resulting $\psi(x,\lambda)$ is the only solution of (6) which lies in $L_2(0,\infty)$. It is therefore reasonable to say that independent of how z varies with b in the lower plane (always $\mathrm{Im}(\lambda) > 0$), the resolvent for L is given by

$$(11)\quad R(\lambda,L)f = \psi(x,\lambda)\int_0^x \phi(y,\lambda)f(y)dy + \phi(x,\lambda)\int_x^\infty \psi(y,\lambda)f(y)dy.$$

That this is actually a resolvent for a closed operator is shown, for instance, by Titchmarsh [5]. The operator is in this case self-adjoint. We can only conclude that the non-realness of the boundary condition does not affect the problem in the limit point case. There is no way around this difficulty if we insist that the operator L have a resolvent for some λ_0 and hence that $\lambda_0 y+y'' - q(x)y = f$ have a solution for all $f \in L_2(0,\infty)$. For any other solution is of the form

$$y = R_0(f) + \text{solution of the homogeneous equation,}$$

where R_0 is given by (11). To satisfy the boundary condition at 0, y must be

of the form $R_0(f) + a\,\phi(x,\lambda_0)$, and this lies in $L_2(0,\infty)$ only if $a = 0$. Thus $R(\lambda_0, L) = R_0$ and L is the above self-adjoint operator. We conclude that the operator L can be defined in only one way in the limit point case* (and this definition will be independent of the boundary condition at infinity), and although it can reasonably be associated with any boundary condition, it is essentially a real-boundary condition operator.

The situation is different in the limit circle case. Here a non-real boundary condition may be associated with a non-self-adjoint operator. For z fixed in say the lower half-plane and $\mathrm{Im}(\lambda) > 0$, the $\ell(\lambda,b,z)$ again form a normal family. However, for a small λ-neighborhood in $\mathrm{Im}(\lambda) < 0$, the $\ell(\lambda,b,z)$ remain outside of the area common to the limit circles for all λ in this neighborhood. Hence again the $\ell(\lambda,b,z)$ form a normal family (on $\mathrm{Im}(\lambda) < 0$ to the closed ℓ-plane). Hence given $b_n \to \infty$ we can find a subsequence b_{n_k} such that $\ell(\lambda, b_{n_k}, z)$ converge uniformly in every compact subset of $[\lambda \mid \mathrm{Im}(\lambda) \neq 0]$ to a function $m(\lambda)$. Note that in the sense of our convergence, infinity may be assumed by $m(\lambda)$ in this domain.

We now examine this process a little more closely. Let us choose three points z_1, z_2, z_3 and obtain a subsequence $\ell(\lambda, b_n, z_i)$ which converges for all three points and $\mathrm{Im}(\lambda) \neq 0$. Several possibilities arise.

1. For some λ_0, say $\mathrm{Im}(\lambda_0) > 0$, all three functions $m(\lambda_0, z_i)$ are distinct. Since they are analytic in $\mathrm{Im}(\lambda) > 0$, they will continue to be distinct except for at most a denumerable set of points having their only points of condensation on the closed real axis.

Now

$$(12) \qquad \frac{\ell(\lambda,z) - \ell(\lambda,z_1)}{\ell(\lambda,z) - \ell(\lambda,z_2)} \cdot \frac{\ell(\lambda,z_3) - \ell(\lambda,z_2)}{\ell(\lambda,z_3) - \ell(\lambda,z_1)} = \frac{z - z_1}{z - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1}$$

and hence in this case $\ell(\lambda, b_n, z)$ converges for all z and all non-exceptional λ $[\mathrm{Im}(\lambda) > 0]$. The limit function $m(\lambda,z)$ satisfies

$$(13) \qquad \frac{m(\lambda,z) - m(\lambda,z_1)}{m(\lambda,z) - m(\lambda,z_2)} \cdot \frac{m(\lambda,z_3) - m(\lambda,z_2)}{m(\lambda,z_3) - m(\lambda,z_1)} = \frac{z - z_1}{z - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1} .$$

In other words $m(\lambda,z)$ is given as a fractional linear transformation of the z-plane onto the closed m-plane (again for non-exceptional λ_n $[\mathrm{Im}(\lambda) > 0]$).

2. For no λ $[\mathrm{Im}(\lambda) \neq 0]$ are the $m(\lambda,z_i)$ distinct. However, for some λ_0 $[\text{say } \mathrm{Im}(\lambda_0) > 0]$, $m(\lambda_0,z_1) \neq m(\lambda_0,z_2) = m(\lambda_0,z_3)$. Again the inequality will persist for all except at most a denumerable set of points having their only points of

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* Note that if $\mathrm{Im}(z) > 0$ the same argument is valid, except that we would limit considerations to $\mathrm{Im}(\lambda) < 0$.

condensation on the closed real axis [again $\mathrm{Im}(\lambda) > 0$]. The equality must persist for all $\mathrm{Im}(\lambda) > 0$; otherwise this case would reduce to case (1). Since the right hand side of (12) remains constant ($\neq 0$) for z different from z_1, z_2, z_3, and since $\ell(\lambda, b_n, z_2) - \ell(\lambda, b_n, z_3) \to 0$, it follows from (12) that $\ell(\lambda, b_n, z) \to m(\lambda, z_2)$ for all $z \neq z_1$ and all non-exceptional λ, $\mathrm{Im}(\lambda) > 0$. Thus the limiting fractional linear transformation is degenerate having the one stable point $m(\lambda, z_2)$ and the one unstable point $m(\lambda, z_1)$. Since all but at most one real z must go into the stable point, the stable point is in any case on the limit circle C_∞. If we define $\psi(x,\lambda)$ and the resolvent as above, the operator corresponding to the stable point will again be self-adjoint.

3. This is the completely degenerate case in which for all choices of z_i we get $m(\lambda, z_1) = m(\lambda, z_2)$, again for $\mathrm{Im}(\lambda) > 0$. In this case the stable and unstable points coincide and the operator corresponding to the stable-unstable point is self-adjoint.

Now case (1) is in a certain sense the exception and will occur in general only for finite and regular boundaries. A regular singular boundary will usually result in cases (2) or (3). Consider, for instance, the case where $x = 0$ is a regular singular boundary.* Here

$$q(x) = a_{-2}/x^2 + a_{-1}/x + a_0 + a_1 x + - - - .$$

The indicial equation is

$$\rho(\rho - 1) - a_{-2} = 0 .$$

Let ρ_1, ρ_2 be the roots of the indicial equation. Let $\rho_1 \geq \rho_2$. Note that $\rho_2 = 1 - \rho_1$. Then two linearly independent solutions of

$$\lambda y + y'' - q(x) y = 0$$

are given by

$$\omega(x) = x^{\rho_1} \left[\omega_0 + \omega_1 x + - - - \right]$$
$$\mu(x) = x^{\rho_2} \left[\mu_0 + \mu_1 x + - - - \right] + c\, \omega(x) \log x$$

where

(a) $\rho_1 - \rho_2 \neq$ integer $\qquad \mu_0 = 1, \qquad c = 0$

(b) $\rho_1 = \rho_2 \qquad \mu_0 = 0, \qquad c = 1$

(c) $\rho_1 - \rho_2 = m > 0 \qquad \mu_0 = 1, \qquad \mu_m = 0$.

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* For convenience we consider the singularity at $x = 0$. The transformation $x' = 1/x$ reduces this to our previous considerations.

In the limit circle case $\int_0^1 |\omega(x)|^2\, dx < \infty$, and $\int_0^1 |\mu(x)|^2\, dx < \infty$. Hence we require that $-\frac{1}{2} < \rho_2 \leq \rho_1$.

Now
$$\theta(x,\lambda) = C_{11}(\lambda)\omega(x,\lambda) + C_{12}(\lambda)\mu(x,\lambda)$$
$$\phi(x,\lambda) = C_{21}(\lambda)\omega(x,\lambda) + C_{22}(\lambda)\mu(x,\lambda)\ .$$

If ρ_1 = integer then so does $\rho_2 = 1 - \rho_1$. But $-\frac{1}{2} < \rho_2 \leq \rho_1$ mean that $\rho_2 = 0$, $\rho_1 = 1$ and hence $a_{-2} = 0$. Hence if $a_{-2} \neq 0$, then for the limit circle case neither ρ_1 nor ρ_2 is integral valued, and it is easily seen that

$$\ell(\lambda,b,z) = -\frac{\theta(b,\lambda)z+\theta'(b,\lambda)}{\phi(b,\lambda)z+\phi'(b,\lambda)} \to -\frac{C_{12}(\lambda)}{C_{22}(\lambda)}$$

and this limit is independent of z.

On the other hand by allowing z to vary as b becomes infinite it is possible to obtain non-self-adjoint operators L even for the degenerate cases (2) and (3). This is accomplished as follows. For a given λ_o (where, say, $\mathrm{Im}(\lambda_o) > 0$) and for $m_o \in C_\infty(\lambda_o)$, there is a unique solution $z(b)$ of

$$m_o = -\frac{\theta(b,\lambda_o)z+\theta'(b,\lambda_o)}{\phi(b,\lambda_o)z+\phi'(b,\lambda_o)}\ .$$

In this case $z(b)$ will lie in the lower half-plane. Choosing a converging sequence of functions $\ell(\lambda,\ b,\ z(b))$, these functions will converge uniformly in every compact subset of $\mathrm{Im}(\lambda) > 0$ to a function $m(\lambda)$ which will be holomorphic for $\mathrm{Im}(\lambda) > 0$. If the $z(b_n)$ should happen to converge to a limit z_o, then it would be reasonable to assign the operator L (which we are about to define) to the boundary condition associated with the point z_o.

We now investigate the operator L with domain D defined as the range of $R(\lambda_o,L)$ given by (11). We note that Eq. (2) is satisfied for $\lambda = \lambda_o$.

Theorem 2.1. The domain D of L consists of all functions $y(x)$ on $(0,\infty)$ such that y and y' are absolutely continuous; y and $-y'' + q(x)y$ belong to $L_2(0,\infty)$; $y(0) = 0$; and $\lim_{x\to\infty} W_x\left[y,\psi(x,\lambda_o)\right] = 0$ where $W_x[u,v] = u(x)v'(x)-u'(x)v(x)$.

Suppose that

$$y(x) = R(\lambda_o,L)f = \psi(x,\lambda_o)\int_0^x \phi(y,\lambda_o)f(y)dy + \phi(x,\lambda_o)\int_x^\infty \psi(y,\lambda_o)f(y)dy,$$

where $f \in L_2(0,\infty)$. Then clearly y and y' are absolutely continuous;

y and $-y'' + q(x)\,y = f$ belong to $L_2(0,\infty)$; and $y(0) = 0$. Finally

$$W_x\,[y,\psi(x,\lambda_0)] = W_x\,[\psi(x,\lambda_0),\psi(x,\lambda_0)] \int_0^x \phi(y,\lambda_0)f(y)dy$$

$$+\ W_x\,[\phi(x,\lambda_0),\psi(x,\lambda_0)] \int_0^\infty \psi(y.\lambda_0)f(y)dy\ ,$$

$$= 0+W_x\,[\phi(x,\lambda_0),\psi(x,\lambda_0)] \int_x^\infty \psi(y,\lambda_0)f(y)dy \to 0,$$

since $W_x\,[\phi(x,\lambda_0),\psi(x,\lambda_0)] \equiv W_0\,[(\phi(x,\lambda_0),\psi(x,\lambda_0)]$ is constant.

Hence if $y \in D$ then y satisfies the conditions of the theorem. Conversely if y satisfies the above conditions, we then set $f(x) = \lambda_0 y + y'' - q(x)y$ (which belongs to $L_2(0,\infty)$ by assumption) and $u(x) = R(\lambda_0,L)f$. Then for $w = y - u$ we have

$$\begin{aligned} \lambda_0 w + w'' - q(x)\,w &= 0 \\ w(0) &= 0 \\ \lim_{x\to\infty} W_x\,[w,\ \psi(x,\lambda_0)] &= 0. \end{aligned} \tag{15}$$

Since $w(0) = 0$ and w satisfies the homogeneous equation, it follows that $w(x) = a\,\phi(x,\lambda_0)$. However

$$W_x\,[w,\ \psi(x,\lambda_0)] = W_x\,[\phi(x,\lambda_0),\ \theta(x,\lambda_0) + m_0\phi(x,\lambda_0)] = c \to 0$$

means that $w(x) \equiv 0$ and hence that y is in the range of $R(\lambda_0,L)$ (that is, in D). This incidentally, proves that $R(\lambda_0,L)$ satisfies Eq. (13). Finally we note that the conditions of the theorem are satisfied by the functions y such that $y(0) = 0$ and y, y', and y'' are absolutely continuous and all vanish outside a finite interval. Hence these functions lie in D; and since they are dense in $L_2(0,\infty)$ then by our definition L is a well-defined operator.

Theorem 2.2. The resolvent $R(\lambda,L)$ exists for all $\mathrm{Im}(\lambda) > 0$ and is given in the usual way by (11).

It is clear that if the range of $R(\lambda,L)$ is contained in D then (2) will be satisfied. Let $f \in L_2(0,\infty)$ and set $y = R(\lambda,L)f$ in Eq. (11). It is again clear that y,y' are absolutely continuous, that y and $-y'' + q(x)y$ belong to $L_2(0,\infty)$, and that $y(0) = 0$. Finally

$$W_x\left[y,\ \psi(x,\lambda_0)\right] = W_x\left[\psi(x,\lambda),\psi(x,\lambda_0)\right] \int_0^x \phi(y,\lambda)\, f(y)\, dy$$

$$+\ W_x\left[\phi(x,\lambda),\psi(x,\lambda_0)\right] \int_x^\infty \psi(y,\lambda)\, f(y)\, dy.$$

Now in the limit circle case both $\int_0^x \phi(y,\lambda)\, f(y)\, dy \rightarrow$ limit and $\int_x^\infty \psi(y,\lambda)\, f(y)\, dy \rightarrow 0$. We now make use of two results due to Titchmarsh [5] (p.22)* namely,

(16) $$\lim_{x\to\infty} W_x\left[\psi(x,\lambda),\ \psi(x,\lambda_0)\right] = 0$$

and ([5] p.19, Eq.(2.1.2)) that $\left|W_x\left[\phi(x,\lambda),\ \psi(x,\lambda_0)\right]\right|$ is bounded; it then follows that $\lim W_x\left[y(x),\ \psi(x,\lambda_0)\right] = 0$. Hence the range of $R(\lambda,L)$ is contained in D. We note that by interchanging the roles of λ_0 and λ, the same argument shows that D is contained in the range of $R(\lambda,L)$ and hence the two are equal. Now suppose $y \in D$. Then it follows that there exists an $f \in L_2(0,\infty)$ such that $y = R(\lambda,L)f$. Hence $(\lambda I - L)y = (\lambda I - L)R(\lambda,L)f = f$; and this proves that $R(\lambda,L)$ satisfies Eq. (3) and is therefore truly the resolvent of L.

For $\operatorname{Im}(\lambda) > 0$, therefore, the resolvent of L exists. Hence $\sigma(L)$, the spectrum of L, is limited to the lower half-plane (here z_0 is in the lower half-plane). We shall, in the appendix, show by example that $\sigma(L)$ need not be limited to a strip parallel to the real axis. We now prove

Theorem 2.3. $\|R(\lambda,L)\| \leq \left[\operatorname{Im}(\lambda)\right]^{-1}$ for $\operatorname{Im}(\lambda) > 0$.

By choice $m(\lambda)$ is a point in $C_\infty(\lambda)$ and hence in $C_b(\lambda)$. Hence there exist two points $\ell(\lambda,b,z_i)$ on the perimeter of $C_b(\lambda)$ (here the z_i are real) such that

(17) $$m(\lambda) = \alpha\,\ell(\lambda,\ b,\ z_1) + \beta\ \ell(\lambda,\ b,\ z_2)$$

where $\alpha,\beta \geq 0$ and $\alpha + \beta = 1$ (fixed λ). Since the operators corresponding to $\ell(\lambda,\ b,\ z_i)$ are self-adjoint on $L_2(0,b)$, the resolvent operators

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* Titchmarsh actually proves only that

$\lim_{n\to\infty} W_{b_n}\left[\psi(x,\lambda),\ \psi(x,\lambda_0)\right] = 0$ where $\ell(\lambda,b_n,z) \rightarrow m(\lambda)$. However since

$W_x\left[\psi(x,\lambda),\ \psi(x,\lambda_0)\right] = W_0\left[\psi(x,\lambda),\ \psi(x,\lambda_0)\right] + (\lambda-\lambda_0)\int_0^x \psi(s,\lambda)\,\psi(s,\lambda_0)ds$ converges to a limit as $x \rightarrow\infty$ in the limit circle case, (16) follows.

$$(18)\qquad R_i(\lambda,L_b)\, f = \left[\,\theta(x,\lambda) + \ell(\lambda,b,z_i)\,\phi(x,\lambda)\,\right] \int_0^x \phi(y,\lambda)\, f(y)\, dy$$
$$+\ \phi(x,\lambda) \int_x^b \left[\,\theta(y,\lambda) + \ell(\lambda,b,z_i)\,\phi(y,\lambda)\right] f(y)\, dy$$

satisfy the inequality

$$\| R_i(\lambda,L_b)\| \leq [\,\mathrm{Im}(\lambda)\,]^{-1}.$$

If $f \in L_2(0,\infty)$ and vanishes for $x > b$, then by (17) and (18) we see that

$$R(\lambda,L)f = \alpha R_1(\lambda,L_b)\, f + \beta R_2(\lambda, L_b)f.$$

Hence

$$\| R(\lambda,L)\, f \| \leq \alpha \| + R_1(\lambda, L_b)f\| + \beta \| R_2(\lambda,L_b)f \| \leq \| f \| \big/ \mathrm{Im}(\lambda).$$

Since this holds for all b, it follows that $\| R(\lambda,L)\| \leq [\mathrm{Im}(\lambda)\,]^{-1}$.

3. Completely continuous resolvents.

A linear bounded operator is said to be completely continuous if it takes bounded sets into compact sets. It frequently happens in problems of the sort we are considering that the resolvent is a completely continuous operator. This is the situation for instance, in the limit circle case or for a self-adjoint operator with a pure point spectrum whose only point of condensation is infinity. We shall now investigate the spectral theory of operators having a completely continuous resolvent. We shall show that such operators have a pure point spectrum whose only point of condensation is infinity, and we shall develop the beginnings of an elementary divisor theory for such operators.

Theorem 3.1. If $R(\lambda_0,L)$ is completely continuous, then so is $R(\lambda,L)$ for all $\lambda \in \rho(L)$.

This follows directly from the first resolvent equation

$$R(\lambda,L) = R(\lambda_0,L) + (\lambda_0 - \lambda)\, R(\lambda_0,L)R(\lambda,L),$$

since the product of a completely continuous operator and a bounded linear operator is completely continuous, and the sum of two completely continuous operators is likewise completely continuous.

Theorem 3.2. If $R(\lambda,L)$ is completely continuous, then L has a pure point spectrum $\{\lambda_n\}$, and if there are infinitely many points in the spectrum, then $\lim|\lambda_n| = \infty$

Let $\lambda_0 \in \rho(L)$. Then by the F.Riesz theory (see Banach, [1], Chap.10) the inverse operator $[\,I + (\lambda-\lambda_0)R(\lambda_0, L)]^{-1}$ exists at all but a set of isolated points $\{\lambda_n\}$. Corresponding to each λ_n there is a finite set of linearly independent vectors x_i^n
$(i = 1, \ldots, k_n)$ which span the null space of $[\,I + (\lambda - \lambda_0)\, R(\lambda_0,L)]$.

Hence

$$(19) \qquad R(\lambda_o,L)\, x_i^n = \frac{1}{\lambda_o - \lambda_n}\, x_i^n \qquad i = 1,2, \ldots, k_n$$

so that

$$x_i^n = (\lambda_o I - L\,)\; R(\lambda_o,L)\, x_i^n = (\lambda_o I - L)\; \frac{x_i^n}{\lambda_o - \lambda_n}\;,$$

and finally

$$(20) \qquad L(x_i^n) = \lambda_n\, x_i^n \qquad i = 1,2,\ldots,k_n.$$

Conversely, if (20) is satisfied then we can retrace our steps and obtain (19). Finally for λ distinct from the λ_n's

$$R(\lambda) = R(\lambda_o,L)\left[I + (\lambda - \lambda_o)\, R(\lambda_o,L)\,\right]^{-1}$$

is easily shown to be the resolvent for L at λ. Hence the points $\{\lambda_n\}$ constitute the point spectrum for L, and all other points belong to the resolvent set of L.

Theorem 3.3. Suppose $R(\lambda,L)$ is completely continuous and that $\lambda_o \epsilon \sigma(L)$. Let M_i be the null space for the operator $(\lambda_o I - L)^i$. Then M_i is of positive finite dimension and there exists a $k > 0$ such that $M_i = M_k$ for all $i > k$, whereas M_i contains M_{i-1} as a proper subset for $i \leq k$.

Since $R(\lambda,L)$ is analytic for $\lambda \epsilon \rho(L)$ (see Hille, [3], Theorem 5.9.2), the integral $\int_C R(\lambda,L)\, d\lambda$ on a rectifiable path C converges in the uniform operator topology and is therefore the limit in this topology of finite sums of completely continuous operators. It follows that

$$E = \frac{1}{2\pi i}\int_C R(\lambda,L)\; d\lambda$$

is itself completely continuous. Since E is also a projection operator (see Hille, [3] Theorem 5.11.1) every bounded subset of the range must be compact, and hence (see Banach, [1] p. 84) the range M of E is finite dimensional. Now let C be a circle about λ_o containing no other points of $\sigma(L)$. For all y, LE exists (see Hille, [3] p.111), and LEy = ELy for all $y \epsilon D$. Thus $D > M$ and LE = EL on M. We set

$$L_o = LE$$

$$R_o(\lambda) = E\, R(\lambda,L) = R(\lambda,L)E\;.$$

Then

$$(\lambda E - L_o)\, R_o(\lambda) = R_o(\lambda)\; (\lambda E - L_o) = E.$$

Thus relative to M, $R_o(\lambda)$ is the resolvent of L_o.

Further, as Hille shows ([3], Theorem 5.10.1), the only singularity of $R_o(\lambda)$ is λ_o. Thus L_o takes M into M and has λ_o as its entire spectrum. Since M is finite dimensional, the conclusions of our theorem are valid for L_o on M, and since λ_o is the only spectral point, $M_k = M$. Suppose that for some $y \in D$ $(\lambda_o - L)y = w \in M$. Then

$$(\lambda_o - \lambda)R(\lambda,L)y + y = R(\lambda,L)\left[(\lambda_o-\lambda)y+(\lambda-L)y\right] = R(\lambda,L)w.$$

$R(\lambda,L)w$ belongs to M and hence

$$y = \frac{1}{2\pi i}\int_C \frac{y}{\lambda-\lambda_o}\,d\lambda = \frac{1}{2\pi i}\int_C R(\lambda,L)y\,d\lambda + \frac{1}{2\pi i}\int_C \frac{R(\lambda,L)}{\lambda-\lambda_o}\,w\,d\lambda$$

$$= E\,y + \text{vector in } M.$$

Thus y itself belongs to M. Consequently if $(\lambda_o I - L)^k y = 0 \in M$, then $(\lambda_o I-L)^{k-1}\,y \in M$ and by induction $y \in M$. Hence all of the subspaces M_i are contained in M and the theorem is then a consequence of its validity for L_o in M.

Corollary. Under the hypothesis of Theorem 3.3, M_k is the range of the projection operator

$$E = \frac{1}{2\pi i}\int_C R(\lambda,L)\,d\lambda$$

where C is a circle about λ_o containing no other spectral points of L.

4. Perturbation theory.

Some differential operators may be treated as perturbations of previously discussed operators. In particular suppose

$$L_o(y) = -y'' + q_o(x)y \qquad 0 \leq x < \infty$$

$$y(0) = 0$$

$$\lim_{x\to\infty}\left[\alpha y(x) + \beta y'(x)\right] = 0$$

where $q_o(x)$ is real valued. Let $q_1(x)$ be bounded and real valued in the region $0 \leq x < \infty$. Then the operator

$$L(y) = -y'' + \left[q_o(x) + i\,q_1(x)\right]y$$

$$y(0) = 0$$

$$\lim_{x\to\infty}\left[\alpha\,y(x) + \beta y'(x)\right] = 0$$

can be treated as the sum of L_o and the bounded linear operator $A(y) = i\,q_1(x)\,y(x)$. Before elaborating on this we prove a relevant theorem.

Theorem 4.1. Let L_0 be a closed linear operator and let A be a bounded linear operator. If $\| R(\lambda_0,L_0)\| < \| A\|^{-1}$ then $R(\lambda_0,L_0 + A)$ exists and $\|R(\lambda_0,L_0+A)\| < \| R(\lambda_0,L_0\| \big/ [1-\| A\| \ \|R(\lambda_0,L_0)\|]$. Further if $R(\lambda,L_0)$ is completely continuous then so is $R(\lambda,L_0 + A)$.

$L_0 + A$ is clearly defined on $D(L_0)$. It is an easy matter to check that $L = L_0 + A$ is linear and closed on $D(L_0)$ which we now take to be the domain of definition D of L as well. We proceed to construct $R(\lambda_0,L)$. Now

$$[\lambda_0 I -(L_0 + A)]\, R(\lambda_0,L_0) = I - A\, R(\lambda_0,L_0).$$

Since

(21a) $||AR(\lambda_0,L_0)|| \leq ||A||\,||R(\lambda_0,L_0)|| < 1,$

it follows that $[I - AR(\lambda_0,L_0)]^{-1}$ exists . In fact

$$(21) \qquad [I - AR(\lambda_0,L_0)]^{-1} = \sum_{n=0}^{\infty} [AR(\lambda_0,L_0)]^n.$$

Hence $R = R(\lambda_0,L_0) \sum_0^{\infty} [AR(\lambda_0,L_0)]^n$ is a right inverse for $\lambda_0 I - L$. Since the range of $[I-AR(\lambda_0,L_0)]^{-1}$ is the entire space, the range of R coincides with the range of $R(\lambda_0,L_0)$ which is D. Thus given $y \in D$ there exists a $w \in L_2(0,\infty)$ such that $y = Rw$. Finally $(\lambda_0 I - L)y = (\lambda_0 I - L)Rw = w$ and hence $R(\lambda_0 I-L)y = Rw = y$. Thus R satisfies Eq. (2) and (3) and is therefore the resolvent of L at λ_0. From (21) we see that

$$(22) \qquad || \, [I-AR(\lambda_0,L_0)]^{-1}|| \leq \sum_0^{\infty} ||\, [AR(\lambda_0,L_0)]^n|| \leq \frac{1}{1-||A|| \ \ ||R(\lambda_0,L_0)||}$$

The required norm inequality follows from (21a).

The fact that R is completely continuous follows from the fact that it is the product of a completely continuous operator $R(\lambda_0,L_0)$ and a bounded operator $[I-AR(\lambda_0,L_0)]^{-1}$.

Corollary 1. If L_0 is self-adjoint and A is bounded linear, then the spectrum of $L = L_0 + A$ is confined to the strip $|\mathrm{Im}(\lambda)| \leq ||A||$, and

$$||R(\lambda,L)|| \leq [\,|\mathrm{Im}(\lambda)|-||A||\,]^{-1} \text{ for } |\mathrm{Im}(\lambda)|>||A||.$$

This is an immediate consequence of the above theorem and the fact that for self-adjoint operators L_0, $|| R(\lambda,L_0)|| \leq |\mathrm{Im}(\lambda)|^{-1}$.

In a similar way we obtain the .

Corollary 2. Let L_o be closed linear and A bounded linear. If, further, the resolvent of L_o exists for $\mathrm{Im}(\lambda) > 0$ and therefore satisfies the inequality $\| R(\lambda, L_o)\| < [\mathrm{Im}(\lambda)]^{-1}$, then the spectrum of $L = L_o + A$ lies in the half plane $\mathrm{Im}(\lambda) \leq \| A \|$ and

$$\| R(\lambda, L)\| \leq [\mathrm{Im}(\lambda) - \| A \|]^{-1} \qquad \text{for } \mathrm{Im}(\lambda) > \| A \| .$$

These corollaries have immediate application to the differential operators which we dealt with in section two. In the limit point case we obtain but one operator, regardless of the boundary condition at infinity, and this operator is self-adjoint. In the limit circle case a variety of operators can be obtained. Those corresponding to boundary conditions z in the lower half-plane have resolvents which satisfy the conditions of corollary 2 by Theorem 2.3.

We close this section with a theorem which shows how the spectrum and the null spaces of I - L vary under small perturbations.

Theorem 4.2. Let L_o be a closed linear operator with a completely continuous resolvent. Let A_ϵ be a one parameter family of bounded linear operators such that $\| A_\epsilon \| \to 0$ as $\epsilon \to 0$. $L_\epsilon = L_o + A_\epsilon$. Let λ_o belong to $\sigma(L_o)$ and set

$$E_\epsilon = \frac{1}{2\pi i} \int_C R(\lambda, L_\epsilon)\, d\lambda$$

where C is a circle about λ_o containing no other point of $\sigma(L_o)$.

Then as $\epsilon \to 0$, C will eventually contain points of $\sigma(L_\epsilon)$, $\| E_\epsilon - E_o \| \to 0$, and $\dim (E_\epsilon) = \dim (E_o)$ for ϵ sufficiently small.

As we have already seen in Theorem 4.1, when $\| A_\epsilon R(\lambda, L_o)\| < 1$,

$$R(\lambda, L_o) - R(\lambda, L_\epsilon) = R(\lambda, L_o) \left[\sum_1^\infty [A_\epsilon R(\lambda, L_o)]^n \right]$$

and hence

$$\| R(\lambda, L_o) - R(\lambda, L_\epsilon)\| < \frac{\| A_\epsilon R(\lambda, L_o)\| \cdot \| R(\lambda, L_o)\|}{1 - \| A_\epsilon R(\lambda, L_o)\|} .$$

Now $R(\lambda, L_o)$ is analytic on C and hence its norm is bounded on C. It follows that for sufficiently small ϵ, $\| A_\epsilon R(\lambda, L_o)\| < 1$ on C and that $\| R(\lambda, L_o) - R(\lambda, L_\epsilon)\| \to 0$ as $\epsilon \to 0$ uniformly for λ on C. It is therefore clear that $\| E_\epsilon - E \| \to 0$ as $\epsilon \to 0$. The dimension statement is an immediate consequence of the following lemma:

Lemma. If E_1, E_2 are finite dimensional projection operators such that $\| E_1 - E_2 \| < 1$, then $\dim (E_1) = \dim (E_2)$.

Proof. Suppose that $\dim (E_1) = n > \dim (E_2)$. Then there exist vectors $x_1, x_2, \ldots, x_n$ which span the range of E_1 such that $\| x_i \| = 1$ and $\| x_k - \sum_{i<k} a_i x_i \| \geq 1$ for all possible choices of the a_i. Now $x_i = E_1(x_i)$. Set $y_i = E_2(x_i)$. Since by assumption $\dim (E_2) < n$, the y_i's will not be linearly independent so that for some k, $y_k = \sum_{i<k} a_i y_i$. Now

$$\begin{aligned} y_k - \sum_{i<k} a_i y_i &= E_2(x_k - \sum_{i<k} a_i x_i) \\ &= E_1(x_k - \sum_{i<k} a_i x_i) + (E_2 - E_1)(x_k - \sum_{i<k} a_i x_i). \end{aligned}$$

Therefore

$$0 = \| y_k - \sum_{i<k} a_i y_i \| \geq \| x_k - \sum_{i<k} a_i x_i \| - \| E_2 - E_1 \| \; \| x_k - \sum_{i<k} a_i x_i \| > 0 ,$$

so that our assumption about $\dim (E_2)$ must have been false. In a similar way we show that $\dim (E_2) \leq \dim (E_1)$.

5. Representation theory.

One of the more useful properties of self-adjoint regular operators is the completeness of the associated set of eigenfunctions. This permits the expansion of arbitrary functions in L_2 in terms of the eigenfunctions. In the singular case, it is possible to represent the functions in L_2 by means of an integral in which wave-packets now play the role of eigenfunctions. In this section we shall make a few beginnings in this direction for the non-self-adjoint operator.

Theorem 5.1. If L is a closed linear operator such that

$\| R(\lambda, L) \| \leq [\operatorname{Im}(\lambda) - \omega]^{-1}$ for $\operatorname{Im}(\lambda) > \omega$, then

$$y = \frac{-1}{\pi i} \int_{i\gamma-\infty}^{i\gamma+\infty} R(\lambda, L) y \, d\lambda$$

for each $y \in D$ and $\gamma > \omega$. For an arbitrary y in L_2 the above integral is Cesaro-summable to y.

If $y \in D$ then $R(\lambda, L)(\lambda I - L) y = y$

and hence

$$R(\lambda, L) y = \frac{y}{\lambda} + \frac{R(\lambda, L) L(y)}{\lambda} .$$

It is clear that $-\frac{1}{\pi i} \int_{i\gamma-\infty}^{i\gamma+\infty} \frac{y}{\lambda} \, d\lambda = y$. Further

$$\int_{i\gamma-a}^{i\gamma+a} \frac{R(\lambda,L)L(y)}{\lambda}\, d\lambda = \int_C \frac{R(\lambda,L)L(y)}{\lambda}\, d\lambda$$

where C is a semi-circle with radius a, center $i\gamma$, and end points $(i\gamma \pm a)$. However

$$\left\| \int_C \frac{R(\lambda,L)L(y)}{\lambda}\, d\lambda \right\| \leq \| L(y) \| \int_0^{\pi} \frac{d\theta}{a \sin\theta \; (\gamma-\omega)}$$

which converges to zero as $a \to \infty$. The fact that the integral is Cesaro-summable to y for arbitrary y follows from the fact that $-iL$ is the infinitesimal generator for a semi-group of linear bounded transformations (see Hille, [3], Chap. 12) and hence the inverse Laplace transform is Cesaro-summable (see Hille, [3] Theorem 11.7.2).

This is as far as we have been able to go in the general case. If we further restrict $R(\lambda,L)$, we can go a bit farther.

Theorem 5.2. If $R(\lambda,L)$ satisfies $\|R(\lambda,L)\| \leq \left[\, |\mathrm{Im}(\lambda)| - \omega \right]^{-1}$ for $|\mathrm{Im}(\lambda)| > \omega$, and if there exist vertical lines ($\mathrm{Re}(\lambda) = a_n$ where $a_n \to \pm\infty$ with n) along which $R(\lambda,L)$ is uniformly bounded in n, then for $y \in D$, $E_n(y) \to y$ where

$$E_n = \frac{1}{2\pi i} \int_{C_n} R(\lambda,L)\, d\lambda$$

and C_n is a rectangular path bounded by $|\mathrm{Re}(\lambda)| = \gamma > \omega$ and $\mathrm{Im}(\lambda) + a_{\pm n}$.

As in the proof of the previous theorem

$$\frac{1}{2\pi i} \left[\int_{-\gamma i + a_{-n}}^{-\gamma i + a_n} + \int_{\gamma i + a_n}^{\gamma i + a_{-n}} R(\lambda,L)y\, d\lambda \right] \to y$$

as $n \to \infty$. On the other hand

$$\int_{a_n - i\gamma}^{a_n + i\gamma} R(\lambda,L)y\, d\lambda = \int_{a_n - i\gamma}^{a_n + i\gamma} \left[\frac{y}{\lambda} + \frac{R(\lambda,L)L(y)}{\lambda} \right] d\lambda$$

which clearly goes to zero as $|n| \to \infty$ because of our boundedness hypothesis on $R(\lambda,L)$.

An application of theorem 5.2 is the following. Suppose L_0 is a self-adjoint operator having gaps in its spectrum of length greater than or equal to b and that these gaps occur outside of every finite interval. Now let A be a bounded linear operator of norm $\|A\| < b/2$. Then along the vertical lines through the midpoint of such a gap, $\|R(\lambda,L_0)\| \leq 2/b$. Hence by Theorem 4.1, $\|R(\lambda,L_0+A)\| \leq \dfrac{2}{b - 2\|A\|}$

along each of these vertical lines. The remaining condition of Theorem 5.2 is satisfied by Corollary 1 to Theorem 4.1. If in addition $R(\lambda, L_0)$ is completely continuous, then again by Theorem 4.1 $R(\lambda, L_0 + A)$ is completely continuous. The above theorem then shows that we can expand an arbitrary $y \in D$ in terms of functions which for some λ and k are in the null space of $\left[\lambda-(L_0+A)\right]^k$.

6. Appendix. We conclude this paper with an example of an operator L for which $\sigma(L)$ is not contained in any strip parallel to the real axis. L is of the form

$$L(y) = x^2 \left[-y'' + y\right] \qquad 1 \leq x < \infty$$

$$(23) \qquad y(1) = 0$$

$$\lim_{x \to \infty} \left[y'(x) - i\, y(x)\right] = 0,$$

and $L_2(1, \infty, \rho)$ has the weight function $\rho = 1/x^2$. By a suitable transformation this operator may be put in the usual form $(-y''+q(x)y)$ on $L_2(0, \infty)$ with weight function 1. However the form (19) is more convenient for our purposes. The solutions of the homogeneous equation

$$(24) \qquad \frac{\lambda}{x^2}\, y + y'' - y = 0$$

are

$$(25) \qquad j_\nu(x) = \sqrt{\frac{\pi x}{2}}\; J_\nu(x), \text{ and } n_\nu(x) = \sqrt{\frac{\pi x}{2}}\; N_\nu(x)$$

where J_ν and N_ν are the Bessel and Neumann functions of order ν, and $\lambda = -(\nu^2 - 1/4)$. These functions are bounded in x and hence belong to $L_2(1, \infty, \rho)$. We are therefore in the limit circle case. As we have seen in section three, this means that we have a pure point spectrum $\{\lambda_n\}$ which may contain a finite or denumerably infinite set of points. In the latter case $|\lambda_n| \to \infty$ with n.

It turns out that

$$(26) \qquad \begin{aligned} \theta(x,\lambda) &= j'_\nu(1)\, n_\nu(x) - n'_\nu(1)\, j_\nu(x) \\ \phi(x,\lambda) &= j_\nu(1)\, n_\nu(x) - n_\nu(1)\, j_\nu(x) \end{aligned}$$

$$m(\lambda) = -\frac{h'_\nu(1)}{h_\nu(1)}$$

$$\psi(x,\lambda) = -\frac{h_\nu(x)}{h_\nu(1)}$$

where

$$h_\nu(x) = j_\nu(x) + i\, n_\nu(x) \equiv \sqrt{\frac{\pi x}{2}}\; H_\nu^{(1)}(x).$$

Hence the spectral points of L are the zeros of $h_\nu(1)$ and therefore the zeros of the Hankel function of the first kind $H_\nu^{(1)}(1)$. We first prove the

Lemma 1. $h_\nu(1)$ has an infinite number of zeros.

Let

$$(27) \qquad u_\lambda(x) = \sqrt{\frac{2}{\pi}}\; e^{-i\frac{\pi}{2}(\nu+1/2)}\, h_\nu(x) \text{ for } \lambda = -(\nu^2 - 1/4).$$

Then $u_\lambda(x)$ behaves like e^{ix} for large x and hence satisfies the integral equation

$$(28) \qquad u_\lambda(x) = e^{ix} + \lambda \int_x^\infty \frac{\sin(x-t)}{t^2}\, u_\lambda(t)\, dt\,.$$

Since $u_\lambda(x)$ is bounded for $x \geq 1$, we obtain by successive approximations the relation

$$(29) \qquad u_\lambda(x) = e^{ix} + \lambda \int_x^\infty \frac{\sin(x-t)}{t^2}\, e^{it}\, dt$$
$$+\lambda^2 \int_x^\infty dt_2\, \frac{\sin(x - t_2)}{t_2^2} \int_{t_2}^\infty dt_1\, \frac{\sin(t_2 - t_1)}{t_1^2}\, e^{it_1} + \ldots$$

Since $|\sin\theta|$ is bounded by one, it is easy to show that

$$|u_\lambda(x) \leq | + \frac{|\lambda(x)|}{1!} + \frac{|\lambda(x)|^2}{2!} + \ldots$$

It now follows that $u_\lambda(1)$ is an entire function in λ of exponential order ≤ 1. Thus if $u_\lambda(1)$ had only a finite number of zeros, Hadamard (see Copson, [2] p. 166) has shown that it must be of the form

$$u_\lambda(1) = e^{a\lambda} \sum_0^N b_n \lambda^n \qquad \text{where } (b_N \neq 0)\,.$$

Now

$$H_\nu(1) = e^{i\pi/2\,(\nu + 1/2)}\; u_{-(\nu^2 - 1/4)}(1)\,,$$

and by a well known difference relation for Bessel functions

$$H_{\nu-1}(x) + H_{\nu+1}(x) = \frac{2\nu}{x} H_\nu(x).$$

Hence

$$-ie^{a(\nu^2 - 2\nu + 3/4)} \sum_0^N h_n(\nu^2 - 2\nu + 3/4)^u$$

$$+ie^{a(\nu^2 + 2\nu + 3/4)} \sum_0^N b_n(\nu^2 + 2\nu + 3/4)^u$$

$$= 2\nu \; e^{a(\nu^2 - 1/4)} \sum_0^N b_n(\nu^2 - 1/4)^n .$$

For $\mathrm{Re}(a) = 0$, we divide through by ν^{2N+1} and let $\nu \to \infty$. This gives $b_N = 0$, contrary to assumption. If $\mathrm{Re}(a) > 0$, we divide through by $\nu^{2N} e^{a(\nu^2 + 2\nu)}$ and pass to the limit as $\nu \to \infty$. This gives $ie^{a^{3/4}} b_N = 0$, again contrary to assumption. Finally if $\mathrm{Re}(a) < 0$, we divide through by $\nu^{2N} \; e^{a(\nu^2 - 2\nu)}$ and pass to the limit, obtaining $-ie^{a^{3/4}} b_N = 0$. Thus in each possible case we obtain a contradiction and hence our assumption that $h_\nu(1)$ has a finite number of zeros must be false.

It follows that $H_\nu^{(1)}(1)$ likewise has an infinite set of zeros $\{\nu_n\}$ and since it is entire in ν, these are isolated points. Thus $|\nu_n| \to \infty$. We now wish to localize these a bit further and to show in particular that $\mathrm{Im}(\lambda_n)$ $\mathrm{Im}(\lambda_n) = \mathrm{Im}\left(-\nu_n^2 + \frac{1}{4}\right) \to -\infty$. By the definition of $u_\lambda(x)$ [see (27)] we have

$$u''_{\lambda_n} + u_{\lambda_n} = -\frac{\lambda_n}{x^2} \; u_{\lambda_n}$$

$$\bar{u}''_{\lambda_n} + \bar{u}_{\lambda_n} = -\frac{\bar{\lambda}_n}{x^2} \; \bar{u}_{\lambda_n}$$

$$u_{\lambda_n}(1) = \bar{u}_{\lambda_n}(1) = 0 .$$

Hence

$$u_{\lambda_n} \; u'_{\lambda_n} - \bar{u}'_{\lambda_n} \; u_{\lambda_n} \Big|_1^{\infty} = -(\lambda_n - \bar{\lambda}_n) \int_1^{\infty} \frac{|u_{\lambda_n}|^2}{t^2} \, dt$$

and therefore

$$2i = -2i \, \mathrm{Im}(\lambda_n) \int_1^{\infty} \frac{|u_{\lambda_n}|^2}{t^2} \, dt .$$

Thus $\mathrm{Im}(\lambda_n) < 0$. This confirms our previous theory that $\sigma(L)$ lies in the lower half-plane if z (here $z = -i$) lies in the lower half-plane.

Now $\mathrm{Im}(\lambda) < 0$ corresponds to the first quadrant in the ν-plane. We shall show that the strips (a) $\mathrm{Re}(\nu) > 0$, $0 \leq \mathrm{Im}(\nu) \leq 1/2$ and (b) $0 \leq \mathrm{Re}(\nu) \leq 1/2$, $\mathrm{Im}(\nu) > 0$ each contain only a finite number of zeros. Translated into the λ-plane this gives the desired result, namely, $\lim \mathrm{Im}(\lambda_n) = -\infty$.

Lemma 2. $H_{\nu}^{(1)}(1)$ has only a finite number of zeros in the strip $\mathrm{Re}(\nu) > 0$, $0 \leq \mathrm{Im}(\nu) \leq 1/2$.

To obtain this result we make use of a relation given by Watson [6], (p. 445)

$$\left| H_{\nu}^{(1)}(x) \right|^2 = \frac{4}{\pi^2} e^{-\beta\pi} \int_0^{\infty} K_{2i\beta}\,(2\,x \sinh t) \cosh\,(2\alpha t + \beta\pi)\, dt$$

where $\nu = \alpha + i\beta$. It is sufficient to show that

$$I_{\alpha\beta} = \int_0^{\infty} K_{2i\beta}\,(2 \sinh t) \cosh(2\alpha t + \beta\pi)\, dt$$

is positive sufficiently far out in the strip (a). By Watson [6], (p.181)

$$K_{2i\beta}(u) = \int_0^{\infty} e^{-u \cosh E} \cos 2\beta t\, dt = \frac{1}{2\beta} \int_0^{\infty} e^{-u \cosh s/2\beta} \cos s\; ds\,.$$

The coefficient of $\cos s$ is monotonic decreasing and hence the tail past $3\pi/2$ is positive. It follows that

$$K_{2i\beta}\,(u) > \frac{1}{2\beta} \int_0^{3\pi/2} e^{-u \cosh s/2\beta} \cos s\, ds.$$

We improve this inequality if we replace $\cosh s/2\beta$ by something larger in $(0,\pi/2)$ and smaller in $(\pi/2, 3\pi/2)$, for instance by $1 + \gamma s$, where $\gamma = \dfrac{\cosh \pi/4\beta - 1}{\pi/2}$.

We then have

$$K_{2i\beta}(u) > \frac{1}{2\beta} \int_0^{3\pi/2} e^{-u - \gamma us} \cos s\, ds$$

$$= \frac{e^{-u}}{2\beta}\left[\frac{\gamma u}{(\gamma u)^2+1} - \frac{e^{-\gamma u\, 3\pi/2}}{(\gamma u)^2+1}\right]$$

$$> -\frac{e^{-u}}{2\beta}\cdot\frac{1}{(\gamma u)^2+1} \qquad \text{for } u < 1/\gamma$$

$$> \frac{1}{2}\ \frac{e^{-u}}{2\beta}\cdot\frac{\gamma u}{(\gamma u)^2+1} \qquad \text{for } u > 2/\gamma.$$

Therefore

$$I_{\alpha\beta} > -\int_0^{1/\gamma} \frac{e^{-u}}{2\beta}\cdot\frac{1}{(\gamma u)^2+1}\cdot\frac{\cosh(2\alpha t+\beta\pi)}{\sqrt{4+u^2}}\,du$$

$$+\frac{1}{2}\int_{2/\gamma}^{\infty} \frac{e^{-u}}{2\beta}\cdot\frac{\gamma u}{(\gamma u)^2+1}\cdot\frac{\cosh(2\alpha t+\beta\pi)}{\sqrt{4+u^2}}\,du$$

where $u = 2\sinh t$. Setting $v = \gamma u = 2\gamma \sinh t$ and $\sinh t_0 = 1/2\gamma$, we get

$$\frac{I_{\alpha\beta}}{\cosh(2\alpha t_0 + B\pi)} > -\int_0^1 \frac{1}{1+v^2}\ \frac{\cosh(2\ t+\beta\pi)}{\cosh(2\ t_0+\beta\pi)}\,dv$$

$$+\frac{1}{2}\int_2^{\infty} e^{-v/\gamma}\ \frac{v}{1+v^2}\ \frac{1}{\sqrt{1+(v/\gamma)^2}}\ \frac{\cosh(2\alpha t+\beta\pi)}{\cosh(2\alpha t_0+\beta\pi)}\,dv.$$

For $0 \le \beta \le 1/2$, $\gamma_0 = \frac{2}{\pi}(\cosh \pi/2 - 1) \le \gamma$. Hence in the strip (a)

$$\frac{2\beta\gamma\ \ I_{\alpha\beta}}{\cosh(2\alpha t_0+\beta\pi)} > -\int_0^1 \frac{1}{1+v^2}\ \frac{\cosh(2\alpha t+\beta\pi)}{\cosh(2\alpha t_0+\beta\pi)}\,dv$$

$$+\ \frac{1}{2}\int_2^{\infty} e^{-v/\gamma_0}\ \frac{v}{1+v^2}\ \frac{1}{\sqrt{1+(v/\gamma_0)^2}}\ \frac{\cosh(2\alpha t+\beta\pi)}{\cosh(2\alpha t_0+\beta\pi)}\,dv.$$

The result now follows from the fact that the first integrand and hence the first integral converges monotonically to zero as $\alpha \to \infty$, whereas the second integrand and hence the second integral converges monotonically to infinity as $\alpha \to \infty$.

Lemma 3. $H_\nu^{(1)}(1)$ has only a finite number of zeros in the strip $0 \le \mathrm{Re}(\nu) \le 1/2$, $0 < \mathrm{Im}(\nu)$.

Here we make use of the relation

$$H_\nu^{(1)}(x) = \frac{J_{-\nu}(x) - e^{-\nu\pi i} J_\nu(x)}{i \sin \nu\pi}$$

and the series expansion for $J_\nu(x)$, viz:

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{\nu+2m}}{m! \, \Gamma(\nu + m + 1)} \quad .$$

For large $|\beta|$, the first term in $J_{\pm\nu}$ dominates the remainder. In fact

$$\frac{\sum_{m>0}}{\text{1st term}} < \sum_{m>0} \frac{\left(\frac{x}{2}\right)^{2m}}{m \, |\beta|^m} < e^{\frac{x^2}{4|\beta|}} -1.$$

For $x = 1$ and $\beta > 1$, this ratio is less than 0.3. Hence it will be sufficient to show that

$$\lim_{\beta \to \infty} \left| \frac{\text{1st term in } J_{-\nu}(1)}{\text{1st term in } e^{-\nu\pi i} \, J_\nu(1)} \right| = 0$$

uniformly in $0 \leq \alpha \leq 1/2$. This ratio is

$$R_{\alpha\beta} = \left| \frac{2^\nu/\Gamma(-\nu+1)}{e^{-\nu\pi i} \, 2^{-\nu}/\Gamma(\nu+1)} \right| = \frac{2^{2\alpha}}{e^{\beta\pi}} \left| \frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} \right| \quad .$$

Now $\left| \frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)} \right| = \frac{\alpha}{(-\alpha)} \prod \frac{1 + (\beta/n+\alpha)^2}{1 + (\beta/n-\alpha)^2} < \frac{\alpha}{(-\alpha)}$

Hence $R_{\alpha\beta} \to 0$ as $\beta \to \infty$ uniformly for $0 < \alpha \leq 1/2$.

In the $\lambda = u + iv$ plane the region corresponding to the strip (a) is the lower half of the parabola $u = - v^2 + 1/2$ and the region corresponding to the strip (b) is the lower half of the parabola $u = v^2$. Since all but a finite set of the zeros of $H_\nu^{(1)}(1)$ lie in the lower half-plane outside of these two regions, it follows that $\lim_{n \to \infty} \operatorname{Im}(\lambda_n) = - \infty$.

References

1. Banach, Stefan, Théorie des Opérations Linéaires, Warsaw, 1932.
2. Copson, E.T., Theory of Functions of a Complex Variable, London, 1935, Oxford Press.
3. Hille, Einar, Functional Analysis and Semi-Groups, American Mathematics Society Colloquium Publication, vol. 31, 1948.
4. Marcuvitz, Nathan, Field Representations in Spherically Stratified Regions, New York University Mathematics Research Group, Research Report No. EM-29.
5. Titchmarsch, E.C., Eigenfunction Expansions associated with Second Order Differential Equations, 1946, Oxford Press.
6. Watson, G.N., Theory of Bessel Functions. Second Edition, 1944, Cambridge Press.
7. Weyl, Herman, Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen. Math. Annalen, 68 (1910), 220-269.

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